

HÖLDER EQUIVALENCE OF COMPLEX ANALYTIC CURVE SINGULARITIES

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ABSTRACT. We prove that if two germs of irreducible complex analytic curves at $0 \in \mathbb{C}^2$ have different sequence of characteristic exponents, then there exists $0 < \alpha < 1$ such that those germs are not α -Hölder homeomorphic. For germs of complex analytic plane curves with several irreducible components we prove that if any two of them are α -Hölder homeomorphic, for all $0 < \alpha < 1$, then there is a correspondence between their branches preserving sequence of characteristic exponents and intersection multiplicity of pair of branches. In particular, we recovery the sequence of characteristic exponents of the branches and intersection multiplicity of pair of branches are Lipschitz invariant of germs of complex analytic plane curves.

1. INTRODUCTION

The recognition problem of embedded topological equivalence of germs of complex analytic plane curves at $0 \in \mathbb{C}^2$ has a complete solution due to K. Brauner, W. Burau, Khäler and O. Zariski (See [2]). For instance, for irreducible germs (branches), it is shown that any two of them are topological equivalent if, and only if, they have the same sequence of characteristic exponents. For germs of complex analytic plane curves with several irreducible components (several branches), it is shown that any two of them are topological equivalent if, and only if, there is a correspondence between their branches preserving sequence of characteristic exponents of branches and intersection multiplicity of pair of branches. In [8], F. Pham and B. Teissier proved that if two germs of complex analytic plane curves at $0 \in \mathbb{C}^2$, let us say X and Y , are topological equivalent as germs embedded in $(\mathbb{C}^2, 0)$ (i.e. there exists a bijection between their branches preserving sequence of characteristic exponents and intersection multiplicity of pair of branches), then there exists a germ of meromorphic bi-Lipschitz homeomorphism

$$\phi: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$$

such that $\phi(X) = Y$. Actually, Pham and Teissier proved the respective converse result exactly as it is stated below. Other versions of this result can be seen in [3] and [6].

Theorem 1.1 (Pham-Teissier). *If there exists a germ of meromorphic bi-Lipschitz homeomorphism $\phi: X \rightarrow Y$ (not necessarily from \mathbb{C}^2 to \mathbb{C}^2), then there exists a correspondence*

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between their branches preserving sequence of characteristic exponents and intersection multiplicity of pair of branches.

In the next, we are going to define the notion of α -Hölder equivalence of germ of subsets in Euclidean spaces, where α is a positive real number. Let us remind that a mapping $f: U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called α -Hölder if there exists a positive real number C such that

$$\|f(p) - f(q)\| \leq C\|p - q\|^\alpha \quad \forall p, q \in U.$$

Definition 1.2. Let (X, x_0) and (Y, y_0) be germs of Euclidean subsets. We say that (X, x_0) is α -Hölder homeomorphic to (Y, y_0) if there exists a germ of homeomorphism $f: (X, x_0) \rightarrow (Y, y_0)$ such that f and its inverse f^{-1} are α -Hölder mappings. In this case, f is called a *bi- α -Hölder homeomorphism* from (X, x_0) onto (Y, y_0) .

Remark 1.3. Let us remark that bi-1-Hölder homeomorphisms from (X, x_0) onto (Y, y_0) are nothing else than bi-Lipschitz homeomorphisms. Moreover, if (X, x_0) is α_0 -Hölder homeomorphic to (Y, y_0) for some $0 < \alpha_0 \leq 1$, then (X, x_0) is α -Hölder homeomorphic to (Y, y_0) for any $0 < \alpha \leq \alpha_0$.

One of the goals of this paper is to prove that for any pair X and Y of germs of irreducible complex analytic curves at $0 \in \mathbb{C}^2$ with different sequence of characteristic exponents, there exists $0 < \alpha < 1$ such X and Y are not α -Hölder homeomorphic. For germs of complex analytic plane curves at $0 \in \mathbb{C}^2$, X and Y , with two branches, we prove that if the contact of their branches are different, then there exists $0 < \alpha < 1$ such X and Y are not α -Hölder homeomorphic. Let us remark that these results generalize Theorem of Pham-Teissier and its versions in [3] and [6], see Corollary 4.8 and 4.10.

2. PRELIMINARIES

Let us begin by establishing some notations. Given two nonnegative functions f and g , we write $f \lesssim g$ if there exists some positive constant C such that $f \leq Cg$. We also denote $f \approx g$ if $f \lesssim g$ and $g \lesssim f$. If f and g are germ of functions on (X, x_0) , we write $f \ll g$ if $g^{-1}(0) \subset f^{-1}(0)$ and $\lim_{x \rightarrow x_0} [f(x)/g(x)] = 0$.

Let Γ_1, Γ_2 be germs of closed subsets at $0 \in \mathbb{R}^n$ such that $\Gamma_1 \cap \Gamma_2 = \{0\}$. For $r > 0$ sufficiently small, let us define

$$f_{\Gamma_1, \Gamma_2}(r) = \inf\{\|\gamma_1 - \gamma_2\| \mid \gamma_i \in \Gamma_i \text{ and } \|\gamma_i\| \geq r; i = 1, 2\}.$$

Definition 2.1. The *contact* of Γ_1 and Γ_2 is the real number below

$$Cont(\Gamma_1, \Gamma_2) = \lim_{r \rightarrow 0^+} \frac{\ln(f_{\Gamma_1, \Gamma_2}(r))}{\ln(r)}.$$

Remark 2.2. Notice that $Cont(\Gamma_1, \Gamma_2)$ is always at least 1 and it may occur $Cont(\Gamma_1, \Gamma_2) = +\infty$.

Proposition 2.3. *Let X and Y be two germs of Euclidean closed subsets at 0 and let $h : (X, 0) \rightarrow (Y, 0)$ be an α -Hölder homeomorphism. If $\Gamma_1, \Gamma_2 \subset X$ are closed, $\Gamma_1 \cap \Gamma_2 = \{0\}$ then*

$$\alpha^2 \text{Cont}(h(\Gamma_1), h(\Gamma_2)) \leq \text{Cont}(\Gamma_1, \Gamma_2) \leq \text{Cont}(h(\Gamma_1), h(\Gamma_2)) \frac{1}{\alpha^2}.$$

Proof. Let $h : (X, 0) \rightarrow (Y, 0)$ be an α -Hölder homeomorphism, in other words, for some positive constant c , we suppose that the homeomorphism h satisfies:

$$\frac{1}{c} \|p - q\|^{\frac{1}{\alpha}} \leq \|h(p) - h(q)\| \leq c \|p - q\|^\alpha \quad \forall p, q \in X.$$

Given $r > 0$ sufficiently small, let us consider $\gamma_i \in \Gamma_i$ ($i = 1, 2$) such that

$$f_{h(\Gamma_1), h(\Gamma_2)}(r) = \|h(\gamma_1) - h(\gamma_2)\|$$

with $\|h(\gamma_1)\| \geq r$ and $\|h(\gamma_2)\| \geq r$. Therefore,

$$\begin{aligned} f_{h(\Gamma_1), h(\Gamma_2)}(r) &= \|h(\gamma_1) - h(\gamma_2)\| \\ &\geq \frac{1}{c} \|\gamma_1 - \gamma_2\|^{\frac{1}{\alpha}} \\ &\geq \frac{1}{c} [f_{\Gamma_1, \Gamma_2}(u)]^{\frac{1}{\alpha}} \text{ where } cu^\alpha = r \end{aligned}$$

and

$$\frac{\ln f_{h(\Gamma_1), h(\Gamma_2)}(r)}{\ln r} \leq \frac{\ln f_{\Gamma_1, \Gamma_2}(u)}{\alpha^2 \ln u + \alpha \ln c} - \frac{\ln c}{\alpha \ln u + \ln c}.$$

Finally, taking $r \rightarrow 0^+$ in the last inequality, we get $\text{Cont}(h(\Gamma_1), h(\Gamma_2)) \leq \frac{1}{\alpha^2} \text{Cont}(\Gamma_1, \Gamma_2)$.

In order to show that $\text{Cont}(\Gamma_1, \Gamma_2) \leq \frac{1}{\alpha^2} \text{Cont}(h(\Gamma_1), h(\Gamma_2))$, we follow a similar way using h^{-1} instead h .

□

3. PLANE BRANCHES

Let C be the germ of an analytically irreducible complex curve at $0 \in \mathbb{C}^2$ (plane branch). We know that, up to an analytic changing of coordinates, one may suppose that C has a parametrization as follows:

$$\begin{aligned} x &= t^n \\ y &= a_1 t^{m_1} + a_2 t^{m_2} + \dots \end{aligned} \tag{1}$$

where $a_1 \neq 0$, n is the multiplicity of C and $y(t) \in \mathbb{C}\{t\}$. In the case that 0 is a singular point of the curve, n does not divide the integer number m_1 .

The series $y(x^{1/n})$ with fractional exponents is known as *Newton-Puiseux parametrization* of C and any other Newton-Puiseux parametrization of C is obtained from the parametrization above via $x^{1/n} \rightarrow wx^{1/n}$ where $w \in \mathbb{C}$ is an n th root of the unit.

Let us denote $\beta_0 = n$ and $\beta_1 = m_1$. Let $e_1 = \gcd(\beta_1, \beta_0)$ be the great common divisor of these two integers. Now, we denote by β_2 the smaller exponent appearing in the series $y(t)$ that is not multiple of e_1 . Let $e_2 = \gcd(e_1, \beta_2)$; and $e_2 < e_1$, and so on. Let us suppose that we have defined $e_i = \gcd(e_{i-1}, \beta_i)$. Thus, we define β_{i+1} as the smaller exponent of the series $y(t)$ that is not multiple of e_i . Since the sequence of positive integers

$$n > e_1 > \dots > e_i > \dots$$

is decreasing, there exists an integer number g such that $e_g = 1$. In this way, we can rewrite Eq. 1 as follows:

$$\begin{aligned} x &= t^n \\ y &= a_{\beta_1} t^{\beta_1} + a_{\beta_1+e_1} t^{\beta_1+e_1} + \dots + a_{\beta_1+k_1 e_1} t^{\beta_1+k_1 e_1} \\ &\quad + a_{\beta_2} t^{\beta_2} + a_{\beta_2+e_2} t^{\beta_2+e_2} + \dots + a_{\beta_q} t^{\beta_q} + a_{\beta_q+e_q} t^{\beta_q+e_q} + \dots \\ &\quad + a_{\beta_g} t^{\beta_g} + a_{\beta_g+1} t^{\beta_g+1} + \dots \end{aligned}$$

where the coefficient of t^{β_i} is nonzero ($1 \leq i \leq g$). Now, we define the integers m_i and n_i via the following equations:

$$\begin{aligned} e_{i-1} &= n_i e_i \\ \beta_i &= m_i e_i \end{aligned}$$

Thus, one may expand y as a fractional power series of x in the following way:

$$\begin{aligned} y(x^{1/n}) &= a_{\beta_1} x^{\frac{m_1}{n_1}} + a_{\beta_1+e_1} x^{\frac{m_1+1}{n_1}} + \dots + a_{\beta_1+k_1 e_1} x^{\frac{m_1+k_1}{n_1}} \\ &\quad + a_{\beta_2} x^{\frac{m_2}{n_1 n_2}} + a_{\beta_2+e_2} x^{\frac{m_2+1}{n_1 n_2}} + \dots + a_{\beta_q} x^{\frac{m_q}{n_1 n_2 \dots n_q}} + a_{\beta_q+e_q} x^{\frac{m_q+1}{n_1 n_2 \dots n_q}} + \dots \\ &\quad + a_{\beta_g} x^{\frac{m_g}{n_1 n_2 \dots n_g}} + a_{\beta_g+e_g} x^{\frac{m_g+1}{n_1 n_2 \dots n_g}} + \dots \end{aligned}$$

The sequence of integers $(\beta_0, \beta_1, \beta_2, \dots, \beta_g)$ is called the *characteristic exponents* of $(C, 0)$, and the sequence $(m_1, n_1), \dots, (m_g, n_g)$ is called the *characteristic pairs* of $(C, 0)$.

Remark 3.1. Any plane branch C with characteristic exponents $(\beta_0, \beta_1, \beta_2, \dots, \beta_g)$, is bi-Lipschitz homeomorphic to another analytic plane branch parametrized in the following way:

$$y(x^{1/n}) = a_{\beta_1} x^{\frac{m_1}{n_1}} + a_{\beta_2} x^{\frac{m_2}{n_1 n_2}} + \dots + a_{\beta_g} x^{\frac{m_g}{n_1 n_2 \dots n_g}}$$

4. MAIN RESULTS

Let us begin this section by stating one of the main results of the paper.

Theorem 4.1. *Let C and \tilde{C} be complex analytic plane branches. If C and \tilde{C} have different sequence of characteristic exponents, then there exists $0 < \alpha < 1$ such that C is not α -Hölder homeomorphic to \tilde{C} . In particular, the branches are not Lipschitz homeomorphic.*

The next example give us an idea how to get a proof of Theorem 4.1.

Example 4.2. Let $C : y^2 = x^5$ and $\tilde{C} : y^2 = x^3$. There is no a bi- $\frac{4}{5}$ -Hölder homeomorphism $F : (C, 0) \rightarrow (\tilde{C}, 0)$.

Proof. Let us suppose that there is a bi- $\frac{4}{5}$ -Hölder homeomorphism $F : (C, 0) \rightarrow (\tilde{C}, 0)$. Let $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ be the following real arcs in $(C, 0)$:

$$\begin{aligned}\Sigma_1 &= \{(r, r^{5/2}) : r \geq 0\}; \\ \Sigma_2 &= \{(ri, r^{5/2} e^{i(5/2)\pi/2}) : r \geq 0\}; \\ \Sigma_3 &= \{(r, -r^{5/2}) : r \geq 0\}; \\ \Sigma_4 &= \{(-ri, r^{5/2} e^{i(5/2)11\pi/2}) : r \geq 0\},\end{aligned}$$

Let us define :

$$\Gamma_k(r) = (re^{i\gamma_k(r)}, r^{3/2} e^{i(3/2)\gamma_k(r)}) \in F(\Sigma_k); \quad k = 1, 2, 3, 4$$

and,

$$r_{k,n} = \|F(\Sigma_k(r_n^{\frac{1}{\alpha}}))\|, \quad \text{para } k = 1, 2 \text{ and } 3.$$

It comes from Proposition 2.3

$$\|\Gamma_1(r_{1,n}) - \Gamma_3(r_{3,n})\| \lesssim (r_n)^{2\alpha^2}.$$

We know that either

$$|\gamma_1(r) - \gamma_2(r)| \leq |\gamma_1(r) - \gamma_3(r)|$$

or

$$|\gamma_1(r) - \gamma_4(r)| \leq |\gamma_1(r) - \gamma_3(r)|,$$

for any $r > 0$. Thus, up to subsequences, one can suppose, for instance, that

$$|\gamma_1(r_{1,n}) - \gamma_2(r_{3,n})| \leq |\gamma_1(r_{1,n}) - \gamma_3(r_{3,n})|, \quad \forall n,$$

hence,

$$\|\Gamma_1(r_{1,n}) - \Gamma_2(r_{3,n})\| \leq \|\Gamma_1(r_{1,n}) - \Gamma_3(r_{3,n})\|, \quad \forall n.$$

By denoting $\delta_j(r_{k,n}) = F^{-1}(\Gamma_j(r_{k,n})); j, k = 1, 2, 3$, we get

$$\delta_1(r_{1,n}) = (h(r_{1,n}), h(r_{1,n})^{5/2})$$

and

$$\delta_2(r_{2,n}) = (ig(r_{2,n}), g(r_{2,n})^{5/2} e^{i(5/2)\pi/2}),$$

where $(r_n)^{\frac{1}{\alpha}} \lesssim |h(r_{1,n})| \approx |g(r_{2,n})| \lesssim (r_n)^\alpha$. Hence,

$$|\delta_1(r_{1,n}) - \delta_2(r_{2,n})| \gtrsim |h(r_{1,n}) - ig(r_{2,n})| \gtrsim (r_n)^{\frac{1}{\alpha}}.$$

Therefore,

$$\begin{aligned}(r_n)^{\frac{1}{\alpha^2}} &\lesssim \|\delta_1(r_{1,n}) - \delta_2(r_{2,n})\|^{\frac{1}{\alpha}} = (f_{\delta_1, \delta_3}(r))^{\frac{1}{\alpha}} \leq \|\delta_1(r_{1,n}) - \delta_2(r_{3,n})\|^{\frac{1}{\alpha}} \\ &= \|F^{-1}(\Gamma_1(r_{1,n})) - F^{-1}(\Gamma_2(r_{3,n}))\|^{\frac{1}{\alpha}} \leq \|\Gamma_1(r_{1,n}) - \Gamma_2(r_{3,n})\| \\ &\leq \|\Gamma_1(r_{1,n}) - \Gamma_3(r_{3,n})\| \lesssim (r_n)^{2\alpha^2}\end{aligned}$$

and, this implies $2\alpha^2 \leq \frac{1}{\alpha^2}$. Then, $\alpha^4 \leq \frac{1}{2} < \frac{4}{5}$, what is a contradiction.

The other cases are analyzed in a completely similar way. \square

In the following, we are going to generalize what was proved in the example above. Let C and \tilde{C} be branches of complex analytic plane curves at $0 \in \mathbb{C}^2$ with the following characteristic pairs $(n_1, m_1), (n_2, m_2), \dots, (n_g, m_g)$ and $(q_1, l_1), (q_2, l_2), \dots, (q_{\tilde{g}}, l_{\tilde{g}})$ respectively. Before the next result, let us define the following rational number

$$k_{ij}(C, \tilde{C}) = \min\left\{\frac{m_j \cdot q_1 \dots q_i + l_i \cdot n_1 \dots n_j}{2 \cdot l_i \cdot n_1 \dots n_j}, \frac{m_j \cdot q_1 \dots q_i + l_i \cdot n_1 \dots n_j}{2 \cdot m_j \cdot q_1 \dots q_i}\right\}. \quad (2)$$

Lemma 4.3. *If $g \neq \tilde{g}$ and α_0 is a positive real number such that*

$$\max\{k_{ij}(C, \tilde{C}); i = \tilde{g} \leq j \leq g \text{ or } j = g \leq i \leq \tilde{g}\} < \alpha_0^4 < 1,$$

then there is not any bi- α -Hölder homeomorphism $F: (C, 0) \rightarrow (\tilde{C}, 0)$, with $\alpha_0 < \alpha < 1$.

Proof. Without loss of generality, we can suppose that $(C, 0)$ and $(\tilde{C}, 0)$ are parametrized as in Remark 3.1 and let us suppose that $g > \tilde{g}$. In this way, we have the following three cases:

- (1) $\frac{l_{\tilde{g}}}{\tilde{n}} \leq \frac{m_j}{n_1 \dots n_j}, \forall \tilde{g} \leq j \leq g$;
- (2) $\frac{m_j}{n_1 \dots n_j} \leq \frac{l_{\tilde{g}}}{\tilde{n}}, \forall \tilde{g} \leq j \leq g$;
- (3) $\exists j_0, \tilde{g} \leq j_0 \leq g$ such that $\frac{m_j}{n_1 \dots n_j} \leq \frac{l_{\tilde{g}}}{\tilde{n}}, \forall \tilde{g} \leq j \leq j_0$ and $\frac{l_{\tilde{g}}}{\tilde{n}} < \frac{m_j}{n_1 \dots n_j}, \forall j_0 < j \leq g$.

Notice that for any of the above cases, there exists at most one index j such that $\frac{l_{\tilde{g}}}{\tilde{n}} = \frac{m_j}{n_1 \dots n_j}$. So, we are going to consider just j such that $\frac{l_{\tilde{g}}}{\tilde{n}} \neq \frac{m_j}{n_1 \dots n_j}$. For instance, let us suppose that $\frac{l_{\tilde{g}}}{\tilde{n}} > \frac{m_j}{n_1 \dots n_j}, \tilde{g} \leq j \leq g$. In this case, let us consider $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ the following arcs on $(\tilde{C}, 0)$:

$$\begin{aligned} \Gamma_1 &= \{(r, b_{\beta_1} r^{\frac{l_1}{q_1}} + b_{\beta_2} r^{\frac{l_2}{q_1 q_2}} + \dots + b_{\beta_{\tilde{g}}} r^{\frac{l_{\tilde{g}}}{\tilde{n}}}) : r \geq 0\}; \\ \Gamma_2 &= \{(ri, b_{\beta_1} r^{\frac{l_1}{q_1}} e^{i(\frac{l_1}{q_1})\frac{\pi}{2}} + b_{\beta_2} r^{\frac{l_2}{q_1 q_2}} e^{i(\frac{l_2}{q_1 q_2})\frac{\pi}{2}} + \dots + b_{\beta_{\tilde{g}}} r^{\frac{l_{\tilde{g}}}{\tilde{n}}} e^{i(\frac{l_{\tilde{g}}}{\tilde{n}})\frac{\pi}{2}}) : r \geq 0\}; \\ \Gamma_3 &= \{(r, b_{\beta_1} r^{\frac{l_1}{q_1}} + b_{\beta_2} r^{\frac{l_2}{q_1 q_2}} + \dots + b_{\beta_{\tilde{g}-1}} r^{\frac{l_{\tilde{g}-1}}{q_1 \dots q_{\tilde{g}-1}}} + b_{\beta_{\tilde{g}}} r^{\frac{l_{\tilde{g}}}{\tilde{n}}} e^{i(\frac{l_{\tilde{g}}}{\tilde{n}})2q_1 \dots q_{\tilde{g}-1}}) : r \geq 0\}; \\ \Gamma_4 &= \{(-ri, b_{\beta_1} r^{\frac{l_1}{q_1}} e^{i(\frac{l_1}{q_1})\left(\frac{4q_1 \dots q_{\tilde{g}-1} + 3}{2}\right)\pi} + b_{\beta_2} r^{\frac{l_2}{q_1 q_2}} e^{i(\frac{l_2}{q_1 q_2})\left(\frac{4q_1 \dots q_{\tilde{g}-1} + 3}{2}\right)\pi} + \dots \\ &\quad + b_{\beta_{\tilde{g}}} r^{\frac{l_{\tilde{g}}}{\tilde{n}}} e^{i(\frac{l_{\tilde{g}}}{\tilde{n}})\left(\frac{4q_1 \dots q_{\tilde{g}-1} + 3}{2}\right)\pi}) : r \geq 0\}. \end{aligned}$$

Let

$$\begin{aligned} \Sigma_k(r) &= (re^{i\sigma_k(r)}, a_{\beta_1} r^{\frac{m_1}{n_1}} e^{i(\frac{m_1}{n_1})\sigma_k(r)} + a_{\beta_2} r^{\frac{m_2}{n_1 n_2}} e^{i(\frac{m_2}{n_1 n_2})\sigma_k(r)} + \dots \\ &\quad + a_{\beta_g} r^{\frac{m_g}{n}} e^{i(\frac{m_g}{n})\sigma_k(r)}) \in F^{-1}(\Gamma_k); \quad k = 1, 2, 3, 4. \end{aligned}$$

At this moment, let us take $\alpha_0 < \alpha < 1$ and suppose that there exists a bi- α -Hölder homeomorphism $F: (C, 0) \rightarrow (\tilde{C}, 0)$.

Let us define

$$r_{k,n} = \|F^{-1}(\Gamma_k(r_n^{\frac{1}{\alpha}}))\|, \text{ for } k = 1, 2 \text{ and } 3.$$

It comes from Proposition 2.3 that:

$$\|\Sigma_1(r_{1,n}) - \Sigma_3(r_{3,n})\| \lesssim (r_n)^{\frac{l_{\tilde{g}}.n_1 \dots n_j + m_j.\tilde{n}}{2.\tilde{n}.n_1 \dots n_j} \alpha^2}$$

Moreover, we know that either

$$|\sigma_1(r) - \sigma_2(r)| \leq |\sigma_1(r) - \sigma_3(r)|$$

or

$$|\sigma_1(r) - \sigma_4(r)| \leq |\sigma_1(r) - \sigma_3(r)|,$$

$\forall r > 0$. Up to a subsequence, we may suppose that

$$|\sigma_1(r_{1,n}) - \sigma_2(r_{3,n})| \leq |\sigma_1(r_{1,n}) - \sigma_3(r_{3,n})|, \forall n.$$

\therefore

$$\|\Sigma_1(r_{1,n}) - \Sigma_2(r_{3,n})\| \leq \|\Sigma_1(r_{1,n}) - \Sigma_3(r_{3,n})\|, \forall n.$$

Now, let us denote $\delta_j(r_{k,n}) = F(\Sigma_j(r_{k,n}))$; $j, k = 1, 2, 3$. Thus,

$$\delta_1(r_{1,n}) = (h(r_{1,n}), b_{\beta_1} h(r_{1,n})^{l_1/q_1} + \dots)$$

and

$$\delta_2(r_{2,n}) = (ig(r_{2,n}), b_{\beta_1} g(r_{2,n})^{l_1/q_1} e^{i(l_1/q_1)\pi/2} + \dots)$$

where

$$(r_n)^{\frac{1}{\alpha}} \lesssim |h(r_{1,n})| \approx |g(r_{2,n})| \lesssim (r_n)^\alpha$$

\therefore

$$|\delta_1(r_{1,n}) - \delta_2(r_{2,n})| \gtrsim |h(r_{1,n}) - ig(r_{2,n})| \gtrsim (r_n)^{\frac{1}{\alpha}}.$$

Hence,

$$\begin{aligned} (r_n)^{\frac{1}{\alpha^2}} &\lesssim \|\delta_1(r_{1,n}) - \delta_2(r_{2,n})\|^{\frac{1}{\alpha}} = (f_{\delta_1, \delta_2}(r))^{\frac{1}{\alpha}} \leq \|\delta_1(r_{1,n}) - \delta_2(r_{3,n})\|^{\frac{1}{\alpha}} \\ &= \|F(\Sigma_1(r_{1,n})) - F(\Sigma_2(r_{3,n}))\|^{\frac{1}{\alpha}} \lesssim \|\Sigma_1(r_{1,n}) - \Sigma_2(r_{3,n})\| \\ &\leq \|\Sigma_1(r_{1,n}) - \Sigma_3(r_{3,n})\| \lesssim (r_n)^{\frac{l_{\tilde{g}}.n_1 \dots n_j + m_j.\tilde{n}}{2.\tilde{n}.n_1 \dots n_j} \alpha^2} \end{aligned}$$

and, this implies

$$\frac{l_{\tilde{g}}.n_1 \dots n_j + m_j.\tilde{n}}{2.\tilde{n}.n_1 \dots n_j} \alpha^2 \leq \frac{1}{\alpha^2}.$$

Then,

$$\alpha^4 \leq \frac{2.\tilde{n}.n_1 \dots n_j}{l_{\tilde{g}}.n_1 \dots n_j + m_j.\tilde{n}} < \frac{l_{\tilde{g}}.n_1 \dots n_j + m_j.\tilde{n}}{2.l_{\tilde{g}}.n_1 \dots n_j},$$

what is a contradiction.

The other cases are analyzed in a completely similar way. □

Lemma 4.4. *Let $(C, 0)$ e $(\tilde{C}, 0)$ be two complex analytic plane branches with $g = \tilde{g}$. If $1 > \alpha_0^4 > k_{ii}(C, \tilde{C}) \neq 1$, $1 \leq i \leq g$, then there is no bi- α -Hölder homeomorphism $F: (C, 0) \rightarrow (\tilde{C}, 0)$, $\forall \alpha_0 < \alpha < 1$.*

Proof. Let $(C, 0)$ and $(\tilde{C}, 0)$ be parametrized as in Remark 3.1. Let $1 \leq j \leq g$ be such that $\frac{l_j n_1 \dots n_j}{m_j q_1 \dots q_j} \neq 1$. Let us suppose that $\frac{l_j n_1 \dots n_j}{m_j q_1 \dots q_j} < 1$, that is $\frac{l_j}{q_1 \dots q_j} < \frac{m_j}{n_1 \dots n_j}$. Let us consider $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ the following arcs in C :

$$\begin{aligned}\Sigma_1 &= \{(r, a_{\beta_1} r^{\frac{m_1}{n_1}} + a_{\beta_2} r^{\frac{m_2}{n_1 n_2}} + \dots + a_{\beta_j} r^{\frac{m_j}{n_1 \dots n_j}} + \dots) : r \geq 0\}; \\ \Sigma_2 &= \{(ri, a_{\beta_1} r^{\frac{m_1}{n_1}} e^{i(\frac{m_1}{n_1})\frac{\pi}{2}} + \dots + a_{\beta_j} r^{\frac{m_j}{n_1 \dots n_j}} e^{i(\frac{m_j}{n_1 \dots n_j})\frac{\pi}{2}} + \dots) : r \geq 0\}; \\ \Sigma_3 &= \{(r, a_{\beta_1} r^{\frac{m_1}{n_1}} + \dots + a_{\beta_{j-1}} r^{\frac{m_{j-1}}{n_1 \dots n_{j-1}}} + a_{\beta_j} r^{\frac{m_j}{n_1 \dots n_j}} e^{i(\frac{m_j}{n_1 \dots n_j})2n_1 \dots n_{j-1}} + \dots) : r \geq 0\}; \\ \Sigma_4 &= \{(-ri, a_{\beta_1} r^{\frac{m_1}{n_1}} e^{i(\frac{m_1}{n_1})\left(\frac{4n_1 \dots n_{g-1}+3}{2}\right)\pi} + a_{\beta_2} r^{\frac{m_2}{n_1 n_2}} e^{i(\frac{m_2}{n_1 n_2})\left(\frac{4n_1 \dots n_{g-1}+3}{2}\right)\pi} + \dots \\ &\quad + a_{\beta_j} r^{\frac{m_j}{n_1 \dots n_j}} e^{i(\frac{m_j}{n_1 \dots n_j})\left(\frac{4n_1 \dots n_{j-1}+3}{2}\right)\pi} + \dots) : r \geq 0\}.\end{aligned}$$

Let

$$\begin{aligned}\Gamma_k(r) &= (re^{i\sigma_k(r)}, b_{\beta_1} r^{\frac{l_1}{q_1}} e^{i(\frac{l_1}{q_1})\sigma_k(r)} + b_{\beta_2} r^{\frac{l_2}{q_1 q_2}} e^{i(\frac{l_2}{q_1 q_2})\sigma_k(r)} + \dots \\ &\quad + b_{\beta_j} r^{\frac{l_j}{q_1 \dots q_j}} e^{i(\frac{l_j}{q_1 \dots q_j})\sigma_k(r)} + \dots) \in F(\Sigma_k); \quad k = 1, 2, 3, 4.\end{aligned}$$

By contradiction, let us suppose that there exists a bi- α -Hölder homeomorphism $F: (C, 0) \rightarrow (\tilde{C}, 0)$, where $\alpha_0 < \alpha < 1$.

In this way, for each n , we define $r_n > 0$ satisfying

$$r_{k,n} = \|F^{-1}(\Gamma_k(r_n^{\frac{1}{\alpha}}))\|, \text{ for } k = 1, 2 \text{ and } 3.$$

It comes from Proposition 2.3 that

$$\|\Gamma_1(r_{1,n}) - \Gamma_3(r_{3,n})\| \lesssim (r_n)^{\frac{l_j \cdot n_1 \dots n_j + m_j \cdot q_1 \dots q_j}{2 \cdot q_1 \dots q_j \cdot n_1 \dots n_j} \alpha^2}$$

Moreover, we know that either

$$|\gamma_1(r) - \gamma_2(r)| \leq |\gamma_1(r) - \gamma_3(r)|$$

or

$$|\gamma_1(r) - \gamma_4(r)| \leq |\gamma_1(r) - \gamma_3(r)|,$$

for all $r > 0$. Up to a subsequence, we may suppose that

$$|\gamma_1(r_{1,n}) - \gamma_2(r_{3,n})| \leq |\gamma_1(r_{1,n}) - \gamma_3(r_{3,n})|, \forall n.$$

\therefore

$$\|\Gamma_1(r_{1,n}) - \Gamma_2(r_{3,n})\| \leq \|\Gamma_1(r_{1,n}) - \Gamma_3(r_{3,n})\|, \forall n.$$

Let us denote $\delta_j(r_{k,n}) = F^{-1}(\Gamma_j(r_{k,n}))$; $j, k = 1, 2, 3$. Thus,

$$\delta_1(r_{1,n}) = (h(r_{1,n}), a_{\beta_1} h(r_{1,n})^{m_1/n_1} + \dots)$$

and

$$\delta_2(r_{2,n}) = (ig(r_{2,n}), a_{\beta_1}g(r_{2,n})^{m_1/n_1} e^{i(m_1/n_1)\pi/2} + \dots)$$

where $(r_n)^{\frac{1}{\alpha}} \lesssim |h(r_{1,n})| \approx |g(r_{2,n})| \lesssim (r_n)^\alpha$. Therefore,

$$|\delta_1(r_{1,n}) - \delta_2(r_{2,n})| \gtrsim |h(r_{1,n}) - ig(r_{2,n})| \gtrsim (r_n)^{\frac{1}{\alpha}}.$$

Hence,

$$\begin{aligned} (r_n)^{\frac{1}{\alpha^2}} &\lesssim \|\delta_1(r_{1,n}) - \delta_2(r_{2,n})\|^{\frac{1}{\alpha}} = (f_{\delta_1, \delta_2}(r))^{\frac{1}{\alpha}} \leq \|\delta_1(r_{1,n}) - \delta_2(r_{3,n})\|^{\frac{1}{\alpha}} \\ &= \|F^{-1}(\Gamma_1(r_{1,n})) - F^{-1}(\Gamma_2(r_{3,n}))\|^{\frac{1}{\alpha}} \lesssim \|\Gamma_1(r_{1,n}) - \Gamma_2(r_{3,n})\| \\ &\leq \|\Gamma_1(r_{1,n}) - \Gamma_3(r_{3,n})\| \lesssim (r_n)^{\frac{l_j \cdot n_1 \dots n_j + m_j \cdot q_1 \dots q_j}{2 \cdot q_1 \dots q_j \cdot n_1 \dots n_j} \alpha^2} \end{aligned}$$

and, therefore,

$$\frac{l_j \cdot n_1 \dots n_j + m_j \cdot q_1 \dots q_j}{2 \cdot q_1 \dots q_j \cdot n_1 \dots n_j} \alpha^2 \leq \frac{1}{\alpha^2},$$

that is

$$\alpha^4 \leq \frac{2 \cdot q_1 \dots q_j \cdot n_1 \dots n_j}{l_j \cdot n_1 \dots n_j + m_j \cdot q_1 \dots q_j} < \frac{l_j \cdot n_1 \dots n_j + m_j \cdot q_1 \dots q_j}{2 \cdot m_j \cdot q_1 \dots q_j},$$

what is a contradiction.

The other cases are similar. □

Proof of Theorem 4.1. Let us suppose by contradiction that C and \tilde{C} are α -Hölder homeomorphic for all $\alpha \in (0, 1)$. From Lemma 4.3, we get $g = \tilde{g}$, and by Lemma 4.4, we know that

$$\frac{l_i}{q_1 \dots q_i} = \frac{m_i}{n_1 \dots n_i}, \forall i.$$

By taking $i = 1$, in the previous equation, we get $\frac{m_1}{n_1} = \frac{l_1}{q_1}$, that is, $n_1 = q_1$ and $m_1 = l_1$. By taking $i = 2$, in the previous equation, we get

$$\frac{m_2}{n_1 n_2} = \frac{l_2}{q_1 q_2}.$$

Since $n_1 = q_1$, it follows that $n_2 = q_2$ and $m_2 = l_2$.

Following in that way, for $i = g$, we get

$$\frac{m_g}{n_1 \dots n_g} = \frac{l_g}{q_1 \dots q_g}.$$

Since we have proved that $n_1 = q_1, n_2 = q_2, \dots, n_{g-1} = q_{g-1}$, we have $n_g = q_g$ and $m_g = l_g$. Then $(m_1, n_1) = (l_1, q_1), (m_2, n_2) = (l_2, q_2), \dots, (m_g, n_g) = (l_g, q_g)$, hence $(C, 0)$ and $(\tilde{C}, 0)$ have the same characteristic exponents. □

In the next, we are dealing with germs of complex analytic plane curves having more than one branch at $0 \in \mathbb{C}^2$ and we are going to arrive in a result like Theorem 4.1. Let us start pointing out the following version of Proposition 2.3 for germs of complex analytic plane curves with several branches.

Proposition 4.5. *Let C and \tilde{C} be germs of complex analytic plane curves at $0 \in \mathbb{C}^2$. Let $h : (C, 0) \rightarrow (\tilde{C}, 0)$ be a bi- α -Hölder homeomorphism. If C_1, \dots, C_r are the irreducible components of C , then $h(C_1), \dots, h(C_r)$ are the irreducible components of \tilde{C} and*

$$\alpha^2 \leq \frac{\text{Cont}(C_i, C_j)}{\text{Cont}(h(C_i), h(C_j))} \leq \frac{1}{\alpha^2}.$$

Proof. By Lemma A.8 in [4], it follows that $h(C_1), \dots, h(C_r)$ are the irreducible components of \tilde{C} and, by Proposition 2.3,

$$\alpha^2 \leq \frac{\text{Cont}(C_i, C_j)}{\text{Cont}(h(C_i), h(C_j))} \leq \frac{1}{\alpha^2}.$$

□

Theorem 4.6. *Let C_1, C_2, \tilde{C}_1 and \tilde{C}_2 be complex analytic plane branches. If $\text{Cont}(C_1, C_2) \neq \text{Cont}(\tilde{C}_1, \tilde{C}_2)$, then there exists $0 < \alpha < 1$ such that $C = C_1 \cup C_2$ is not α -Hölder homeomorphic to $\tilde{C} = \tilde{C}_1 \cup \tilde{C}_2$.*

Proof. Let us take

$$\alpha_0^2 = \min \left\{ \frac{\text{Cont}(\tilde{C}_1, \tilde{C}_2)}{\text{Cont}(C_1, C_2)}, \frac{\text{Cont}(C_1, C_2)}{\text{Cont}(\tilde{C}_1, \tilde{C}_2)} \right\} < 1.$$

So, it comes from Proposition 4.5 that C is not α -Hölder homeomorphic to \tilde{C} with $\alpha_0 < \alpha < 1$. □

As a consequence of Theorems 4.1 and 4.6, we get the following

Theorem 4.7. *Let C and \tilde{C} be germs of complex analytic plane curves at $0 \in \mathbb{C}^2$. Let C_1, \dots, C_r and $\tilde{C}_1, \dots, \tilde{C}_s$ be the branches of C and \tilde{C} , respectively. If, for each $\alpha \in (0, 1)$, there exists a bi- α -Hölder homeomorphism between C and \tilde{C} , then there is a bijection $\sigma : \{1, \dots, r\} \rightarrow \{1, \dots, s\}$ such that*

- i) *the branches C_i and $\tilde{C}_{\sigma(i)}$ have the same characteristic exponents, for $i = 1, \dots, r$;*
- ii) *the pair of branches (C_i, C_j) and $(\tilde{C}_{\sigma(i)}, \tilde{C}_{\sigma(j)})$ have the same intersection multiplicity at 0, for $i \neq j \in \{1, \dots, r\}$.*

Proof. Let us remark that, if $h : C \rightarrow \tilde{C}$ is a homeomorphism, then by Lemma A.8 in [4], as already used in the proof of the Theorem 4.1, for each $u \in \{1, \dots, r\}$ there is exactly one $j \in \{1, \dots, s\}$ such that $h(C_u) = \tilde{C}_v$ and, in particular, $r = s$. Let $E = \{\frac{1}{2}\} \cup \{k = k_{ij}(C_u, \tilde{C}_v); u, v \in \{1, \dots, r\} \text{ and } k < 1\} \cup \{k = \min \left\{ \frac{\text{Cont}(\tilde{C}_u, \tilde{C}_v)}{\text{Cont}(C_i, C_j)}, \frac{\text{Cont}(C_i, C_j)}{\text{Cont}(\tilde{C}_u, \tilde{C}_v)} \right\}; i, j, u, v \in \{1, \dots, r\}, i \neq j, u \neq v \text{ and } k < 1\}$. We have that E is a finite and non-empty set with $k_0 = \max E < 1$. Thus, let $\alpha \in (0, 1)$ such that $\alpha^4 > k_0$ and let $h : C \rightarrow \tilde{C}$ be a bi- α -Hölder homeomorphism. By Theorem 4.1, for each $i \in \{1, \dots, r\}$, C_i and $\tilde{C}_{\sigma(i)} = h(C_i)$ have the same characteristic exponents. Moreover, for each $i, j \in \{1, \dots, r\}$, by Theorem 4.6, $\text{Cont}(C_i, C_j) = \text{Cont}(\tilde{C}_{\sigma(i)}, \tilde{C}_{\sigma(j)})$. Since $\text{Cont}(C_i, C_j) = \text{Cont}(\tilde{C}_{\sigma(i)}, \tilde{C}_{\sigma(j)})$, it comes from Lemma 3.1 in [3] that the pairs (C_i, C_j) and $(\tilde{C}_{\sigma(i)}, \tilde{C}_{\sigma(j)})$ have the same coincidence at 0 and, therefore, by Proposition 2.4 in [5], we get that the pairs (C_i, C_j) and $(\tilde{C}_{\sigma(i)}, \tilde{C}_{\sigma(j)})$ have the same intersection multiplicity at 0. □

We are going to show that Theorem 4.7 generalizes some known results which we list below. For instance, since Lipschitz maps are α -Hölder for all $0 < \alpha \leq 1$, we obtain, as a first application of Theorem 4.7, the main result in [3].

Corollary 4.8. *Let X and Y be germs of complex analytic plane curves at $0 \in \mathbb{C}^2$. If there exists a bi-Lipschitz subanalytic map between X and Y , then X and Y are topologically equivalent.*

Actually, we do not use the subanalytic hypotheses in Theorem 4.7, hence we obtain the following result proved in [6].

Corollary 4.9. *Let X and Y be germs of complex analytic plane curves at $0 \in \mathbb{C}^2$. If there exists a bi-Lipschitz homeomorphism between X and Y , then X and Y are topologically equivalent.*

Since germs of complex analytic curves in \mathbb{C}^n (spatial curves) are bi-Lipschitz homeomorphic to their generic projections, we also get, as a immediate consequence of Theorem 4.7 the following.

Corollary 4.10. *Let X and Y be germs of complex analytic curves in \mathbb{C}^n and \mathbb{C}^m respectively. If there exists a bi- α -Hölder homeomorphism between X and Y , for all $0 < \alpha \leq 1$, then X and Y are bi-Lipschitz homeomorphic.*

Proof. Let $\tilde{X} \subset \mathbb{C}^2$ and $\tilde{Y} \subset \mathbb{C}^2$ be generic projections of X and Y respectively. Since X and \tilde{X} (respectively Y and \tilde{Y}) are bi-Lipschitz homeomorphic, it follows that \tilde{X} and \tilde{Y} are α -Hölder homeomorphic for all $\alpha \in (0, 1)$. By Theorem 4.7, there exist a bijection between the branches of \tilde{X} and \tilde{Y} that preserves characteristic exponents of branches and, also, preserves intersection multiplicity of pairs of branches. Hence, using Pham-Teissier Theorem (quoted in the introduction), \tilde{X} and \tilde{Y} come bi-Lipschitz homeomorphic. It finishes the proof. \square

We also obtain, in the case of complex analytic plane curves, a generalization of the main result in [1] and the Theorem 4.2 in [7].

Corollary 4.11. *Let $X \in \mathbb{C}^n$ be a germ of complex analytic curve at the origin. Suppose that, for each $\alpha \in (0, 1)$, there is a bi- α -Hölder homeomorphism $h : (X, 0) \rightarrow (\mathbb{C}, 0)$. Then, $(X, 0)$ is smooth.*

We would like to finish this section by stressing the existence of germ of sets that are α -Hölder homeomorphic, for all $0 < \alpha < 1$, but are not bi-Lipschitz homeomorphic.

Definition 4.12. We say that $h : C \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a *log-Lipschitz map*, if there exists $K > 0$ such that $\|h(x) - h(y)\| \leq K\|x - y\| \cdot |\log\|x - y\||$, whenever $x, y \in C$ and $\|x - y\| < 1$.

Remark 4.13. If h is a log-Lipschitz map, then h is α -Hölder, for all $\alpha \in (0, 1)$.

Definition 4.14. Let (X, x_0) and (Y, y_0) be germs of Euclidean subsets. We say that (X, x_0) is *bi-log-Lipschitz homeomorphic* to (Y, y_0) if there exists a germ of homeomorphism $f: (X, x_0) \rightarrow (Y, y_0)$ such that f and its inverse f^{-1} are log-Lipschitz mappings. In this case, f is called a *bi-log-Lipschitz homeomorphism* from (X, x_0) onto (Y, y_0) .

Corollary 4.15. Let C and \tilde{C} be germs of complex analytic plane curves at $0 \in \mathbb{C}^2$. If C and \tilde{C} are bi-log-Lipschitz homeomorphic, then they are bi-Lipschitz homeomorphic.

According to the example below, one see that the last corollary is very dependent on the rigidity of analytic complex structure of the sets.

Example 4.16. Let $\tilde{C} = \{(x, y) \in \mathbb{R}^2; y = |x \log|x||\} \cup \{(0, 0)\}$. The homeomorphism $h: (\mathbb{R}, 0) \rightarrow (\tilde{C}, 0)$ given by $h(x) = (x, |x \log|x||)$ ($h(0) = (0, 0)$) is a bi-log-Lipschitz homeomorphism. However, $(\tilde{C}, 0)$ is not bi-Lipschitz homeomorphic to $(\mathbb{R}, 0)$.

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